Exercise Sheet for *Topology I*, 2017/18

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Sheet 6 due Wednesday, November 29th, 2017

ABCDEFGHIJKLMNOPQRSTUVWXYZ

Exercise 22 (The Alphabet) Consider each letter of the Alphabet as a subspace of $\mathbb{R}^2$.

Which spaces are homeomorphic and which are not?

Determine the homotopy classes of letters. You may use $\pi_1(S^1) \cong \mathbb{Z}$.

Exercise 23 (Cofinite Topology)

Given a set X. Proof that

$$\mathcal{O} := \{U \subset X \,|\, |X \backslash U| < \infty\}$$

provides X with the coarsest topology, such that X is T_1 (points are closed).

Exercise 24 (Quotient of a Quotient)

Given a topological space X and given relations $R_1, R_2 \subset X \times X$.

1. Proof that the quotient map $X \to X/R_i$ induces a map $f_i\colon X \times X \to X/R_i \times X/R_i$ for $i = 1, 2$.
2. Remark that $f_i(R_j)$ defines a relation on X/R_i for $i, j = 1, 2$.
3. We define

$$(X/R_i)/R_j := (X/R_i)/f_i(R_j).$$

   Proof that $(X/R_1)/R_2 \cong (X/R_2)/R_1$.

Exercise 25 (Product of a Quotient vs Quotient of a Product)

We have seen, that two quotients commute. Proof that a product of a quotient, needs not to be a quotient, i.e. the map

$$p \times \mathrm{id} := \mathbb{R}_{>0} \times \mathbb{Q} \to \mathbb{R}_{>0}/\mathbb{Z}_{>0} \times \mathbb{Q}, \quad (x, q) \mapsto ([x], q)$$

is not a quotient map:

1. Proof that is suffices to construct a set $V \subset \mathbb{R}_{>0}/\mathbb{Z}_{>0} \times \mathbb{Q}$ such that $(p \times \mathrm{id})^{-1}(U)$ is open in $\mathbb{R}_{>0} \times \mathbb{Q}$ but V is not open with respect to the product topology.
2. Show that
$$U_n := \{(x, y) \in (n - \frac{1}{4}, n + \frac{1}{4}) \times \mathbb{Q}, \text{ such that } |x - n| < |y - \frac{\sqrt{2}}{n}|\} \subset \mathbb{R}_{>0} \times \mathbb{Q}$$
is open.
3. Show that $U_n \cap (\{n\} \times \mathbb{Q}) = \{n\} \times \mathbb{Q}$.
4. Show that $(p \times \mathrm{id})^{-1}(p \times \mathrm{id})(\bigcup_{i=1}^{\infty} U_n) = \bigcup_{i=1}^{\infty} U_n$. Conclude that it suffices to show that $V := (p \times \mathrm{id})(\bigcup U_n)$ is not open.
5. Show that $([1], 0) \in U$, but $([1], 0) \notin U^{\circ}$.