Exercise Sheet for *Topology I*, 2017/18

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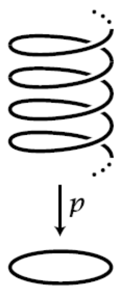
Sheet 7 due Wednesday, December 13th, 2017



The universal cover of a circle.

Exercise 26 (Fundamental group of a wedge) Given spaces with base points (X, x_0) and (Y, y_0). Their wedge

$$X \vee Y := X \times \{0\} \cup Y \times \{1\}/(x_0, 0) \sim (y_0, 1)$$

is the space obtained by taking the union and gluing together the base points. This base has a canonical base point $(x_0, 0) = (y_0, 1)$. Given a map $f\colon X_1 \to X_2$ there is a canonical way of constructing $f \vee \mathrm{id}\colon X_1 \vee Y \to X_2 \vee Y$.

A different way to view $X \vee Y$ is as a subspace of the product $X \times Y$ namely as $\{(x, y) \in X \times Y | x = x_0 \vee y = y_0$.

1. Show that the inclusion $i\colon X \vee Y \to X \times Y$ induces a surjective map of fundamental groups. (Use the isomorphism of $\pi_1(X \times Y)$. Retracts can also be useful.)
2. Show explicitely that the map $\pi_1(i)$ abelisizes $\pi_1(S^1 \vee S^1)$. I.e. given an element $\pi_1(S^1 \vee \mathrm{pt})$ and one in $\pi_1(\mathrm{pt} \vee S^1)$, we include them into $\pi_1(S^1 \times S^1)$, but they might not commute ($[f] \cdot [g] \neq [g] \cdot [f]$). Show that they will commute after applying $\pi_1(i)$ by constructing an explicit homotopy between $\pi_1(i)([f] \cdot [g])$ and $\pi_1(i)([g] \cdot [f])$.

Exercise 27 (Sorgenfrey line and plane) Consider the space $\mathbb{R}_l$ consisting of the real line with the topology generated by all $[a, b)$ being open for all $a, b \in \mathbb{R}$.

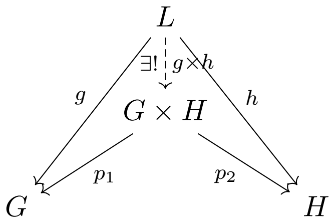
1. Show that the topology on $\mathbb{R}_l$ conincides with the topology incuced by $(-\infty, a)$ and $[b, \infty)$ closed for all $a, b \in \mathbb{R}$.
2. Show that the subspace topology of $\mathbb{R}_l$ on $\mathbb{Q}$ differs from the euclidian topology.
3. Show that $\mathbb{R}_l$ is totally disconnected.
4. Show that $\mathbb{R}$ is normal, i.e. any two disjoint closed sets have disjoint open neighborhoods. (For this it is useful to understand the closed sets well.)
5. Consider $\mathbb{R}_l^2 := \mathbb{R}_l \times \mathbb{R}_l$ with the product topology. Show that the antidiagonal $D := \{(x, -x) | x \in \mathbb{R}\} \subset \mathbb{R}_l^2$ is a discrete subset, i.e. the subspace topology is the discrete topology.
6. Show that $\mathbb{R}_l^2$ is not normal. (Consider $D \cap \mathbb{Q}_l^2$ and $D \backslash \mathbb{Q}_l^2$.)

Exercise 28 ((Co-)Product of groups) Let G, H be groups. Then $G \times H = \{(g, h) \,|\, g \in G\, h \in H\}$ together with the multiplication

$$(g, h) \cdot (g', h') = (gg', hh')$$

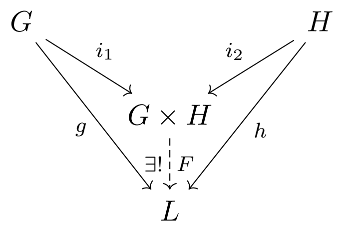
is again a group.

1. Proof that $G \times H$ has the universal property of a product, i.e. for any group L with group homorphisms $g\colon L \to G$ and $h\colon L \to H$ there exists a unique group homorphism $g \times h\colon L \to G \times H$ that makes the diagram commute:



   Here p_1, p_2 denote the projections.

2. Suppose G, H are abelian groups. Proof that $G \times H$ is also the product in the category of abelian groups (i.e. any L is required to be abelian).
3. Suppose G, H are abelian groups. Proof that $G \times H$ is also the coproduct or sum, i.e. for any abelian group L with group homorphisms $g\colon G \to L$ and $h\colon H \to L$ there exists a unique group homorphism $F\colon G \times H \to L$ that makes the diagram commute:



4. Suppose G, H are not necessarily abelian groups. Proof that $G \times H$ is not the coproduct in the category of groups.