Exercise Sheet for *Topology I*, 2017/18

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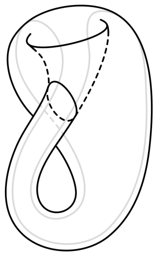
Sheet 8 due Wednesday, December 20th, 2017



The Klein bottle. This is the space $[0,1]^2/\sim$ where $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$.

Exercise 29 (Compact-open topology and induced functions) Let X, Y, Z be topological spaces and let $f\colon Y \to Z$ be continuous. Prove that the induced maps:

$$\mathrm{Hom}(X,Y) \xrightarrow{f_*} \mathrm{Hom}(X,Z), \quad g \mapsto f \circ g$$
$$\mathrm{Hom}(Z,X) \xrightarrow{f^*} \mathrm{Hom}(Y,X), \quad g \mapsto g \circ f$$

are continuous with respect to the compact-open topology.

Exercise 30 (Product vs. Sum for Abelian groups) Let $(A_i)_{i\in I}$ be a family of non-trivial abelian groups. We have seen that for finite families $\prod_{i\in I} A_i$ is also the coproduct. This is not true in general.

We define

$$\prod_{i\in I} A_i = \{(a_i)_{i\in I},\, a_i \in A\}$$

with induced addition $(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}$. On the other hand we define

$$\bigoplus_{i\in I} A_i = \{\sum_{i\in I} a_i \,|\, \text{all but finitely many } a_i \text{ are zero}\}$$

here $\sum_{i\in I} a_i + \sum_{i\in I} b_i = \sum_{i\in I}(a_i + b_i)$.

1. Show that there is a canonical inclusion $\bigoplus_{i\in I} A_i \to \prod_{i\in I} A_i$ that is an isomorphism if and only if I is finite.
2. Show that $\prod_{i\in I} A_i$ together with the canonical projections $p_i\colon \prod_{k\in I} A_k \to A_i$ is a correct definition for the product, i.e. let B be an abelian group with maps $f_i\colon B \to A_i$ than there exists a unique map $g\colon B \to \prod_{i\in I} A_i$ such that for all $i \in I$ we have $p_i \circ g = f_i$.
3. Show that $\bigoplus_{i\in I} A_i$ together with the canonical inclusions $j_i\colon A_i \to \bigoplus_{k\in I} A_k$ is the sum, i.e. given an abelian group C together with maps $f_i\colon A_i \to C$ then there exists a uniqe map $g\colon \bigoplus_{i\in I} A_i \to C$ such that $g \circ j_i = f_i$ for all $i \in I$.

Hence we have shown, that only finite coproducts and products are the same for abelian groups.

Remark: All maps in this exercise are understood to be group homomorphisms.

Exercise 31 (Compactness and Mapping space) Consider the space $I^I = \mathrm{Hom}(I, I)$ with the compact-open topology, where $I = [0, 1]$. As we know I is compact. Prove that I^I is not compact.

(If necessary you may search the web for more information on the compact-open topology.)

Exercise 32 ((Co-)Product of groups continued) Recall that any group can be written as a set of generators along with relations. For example

$$\mathbb{Z} \cong \langle a \rangle, \qquad \mathbb{Z} \times \mathbb{Z} \cong \langle a, b | aba^{-1}b^{-1} = 1 \rangle.$$

Relations can be arbitrary difficult and there is no algorithm that can decide in general wether or not a group is trivial if it is given by relations. However, some groups are easily understood, if given by relations. E.g the symmetry group of an n-gone is given by

$$\langle s, t | s^2 = t^n = (st)^2 = 1 \rangle.$$

Suppose that G, H are (not necessarily) abelian groups. Consider the following group

$$\begin{aligned} G * H := \langle g \in G, h \in H | g_1 \cdot_{G*H} g_2 = (g_2 \cdot_G g_2), g_1, g_2 \in G, \\ h_1 \cdot_{G*H} h_2 = (h_2 \cdot_G h_2), \ h_1, h_2 \in H \rangle. \end{aligned}$$

Here the only relations are given by the relations of G and H. This is called the free product of G and H not to be confused with the free abelian product $G \times H$ for G and H abelian.

1. Describe $\mathbb{Z} * \mathbb{Z}$.
2. Show that $G * H$ is the coproduct in the category of groups.

At some point we will see that $\pi_1(S^1 \vee S^1, x_0) \cong \pi_1(S^1, x_0) * \pi_1(S^1, x_0) \cong \mathbb{Z} * \mathbb{Z}$. Even more: This is true for two arbitrary nice spaces.