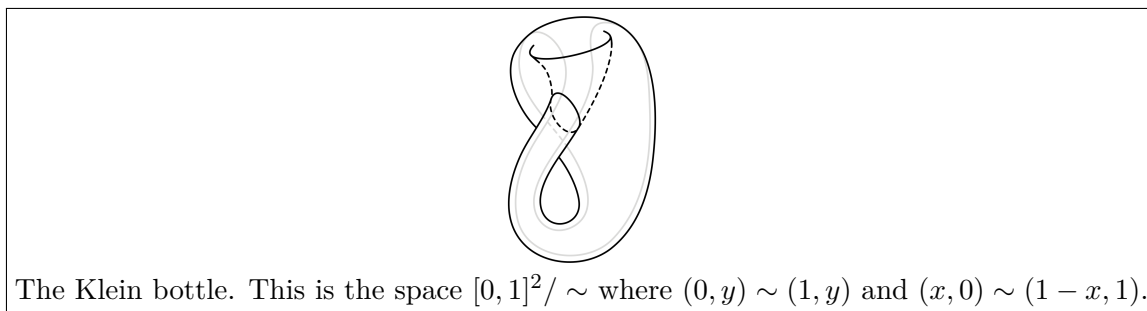


# Exercise Sheet for *Topology I*, 2017/18

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Sheet 8

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**Exercise 29** (Compact-open topology and induced functions) Let  $X, Y, Z$  be topological spaces and let  $f: Y \rightarrow Z$  be continuous. Prove that the induced maps:

$$\begin{aligned} \text{Hom}(X, Y) &\xrightarrow{f_*} \text{Hom}(X, Z), & g &\mapsto f \circ g \\ \text{Hom}(Z, X) &\xrightarrow{f^*} \text{Hom}(Y, X), & g &\mapsto g \circ f \end{aligned}$$

are continuous with respect to the compact-open topology.

**Exercise 30** (Product vs. Sum for Abelian groups) Let  $(A_i)_{i \in I}$  be a family of non-trivial abelian groups. We have seen that for finite families  $\prod_{i \in I} A_i$  is also the coproduct. This is not true in general.

We define

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I}, a_i \in A_i\}$$

with induced addition  $(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I}$ . On the other hand we define

$$\bigoplus_{i \in I} A_i = \left\{ \sum_{i \in I} a_i \mid \text{all but finitely many } a_i \text{ are zero} \right\}$$

here  $\sum_{i \in I} a_i + \sum_{i \in I} b_i = \sum_{i \in I} (a_i + b_i)$ .

1. Show that there is a canonical inclusion  $\bigoplus_{i \in I} A_i \rightarrow \prod_{i \in I} A_i$  that is an isomorphism if and only if  $I$  is finite.
2. Show that  $\prod_{i \in I} A_i$  together with the canonical projections  $p_i: \prod_{i \in I} A_i \rightarrow A_i$  is a correct definition for the product, i.e. let  $B$  be an abelian group with maps  $f_i: B \rightarrow A_i$  then there exists a unique map  $g: B \rightarrow \prod_{i \in I} A_i$  such that for all  $i \in I$  we have  $p_i \circ g = f_i$ .
3. Show that  $\bigoplus_{i \in I} A_i$  together with the canonical inclusions  $j_i: A_i \rightarrow \bigoplus_{i \in I} A_i$  is the sum, i.e. given an abelian group  $C$  together with maps  $f_i: A_i \rightarrow C$  then there exists a unique map  $g: \bigoplus_{i \in I} A_i \rightarrow C$  such that  $g \circ j_i = f_i$  for all  $i \in I$ .

Hence we have shown, that only finite coproducts and products are the same for abelian groups.

Remark: All maps in this exercise are understood to be group homomorphisms.

**Exercise 31** (Compactness and Mapping space) Consider the space  $I^I = \text{Hom}(I, I)$  with the compact-open topology, where  $I = [0, 1]$ . As we know  $I$  is compact. Prove that  $I^I$  is not compact. (If necessary you may search the web for more information on the compact-open topology.)

**Exercise 32** ((Co-)Product of groups continued) Recall that any group can be written as a set of generators along with relations. For example

$$\mathbb{Z} \cong \langle a \rangle, \quad \mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle.$$

Relations can be arbitrary difficult and there is no algorithm that can decide in general whether or not a group is trivial if it is given by relations. However, some groups are easily understood, if given by relations. E.g the symmetry group of an  $n$ -gone is given by

$$\langle s, t \mid s^2 = t^n = (st)^2 = 1 \rangle.$$

Suppose that  $G, H$  are (not necessarily) abelian groups. Consider the following group

$$G * H := \langle g \in G, h \in H \mid g_1 \cdot_{G*H} g_2 = (g_2 \cdot_G g_1), g_1, g_2 \in G, \\ h_1 \cdot_{G*H} h_2 = (h_2 \cdot_G h_1), h_1, h_2 \in H \rangle.$$

Here the only relations are given by the relations of  $G$  and  $H$ . This is called the free product of  $G$  and  $H$  not to be confused with the free abelian product  $G \times H$  for  $G$  and  $H$  abelian.

1. Describe  $\mathbb{Z} * \mathbb{Z}$ .
2. Show that  $G * H$  is the coproduct in the category of groups.

At some point we will see that  $\pi_1(S^1 \vee S^1, x_0) \cong \pi_1(S^1, x_0) * \pi_1(S^1, x_0) \cong \mathbb{Z} * \mathbb{Z}$ . Even more: This is true for two arbitrary nice spaces.