Exercise Sheet for *Topology I*, 2017/18

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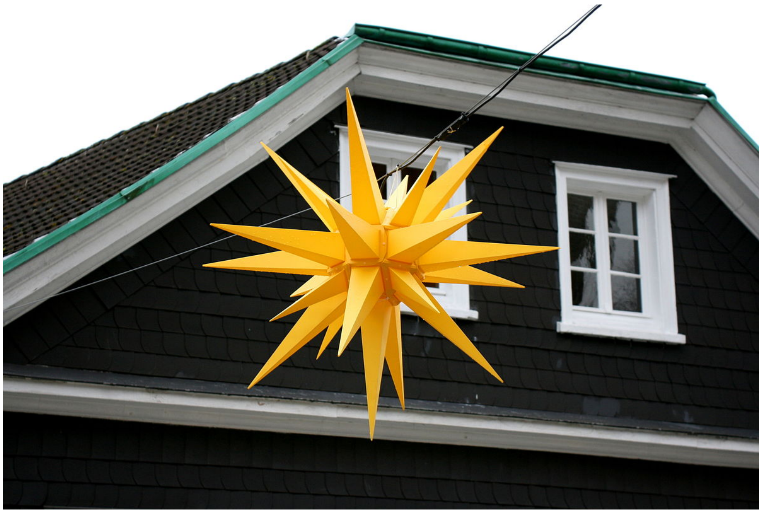
Sheet 9 due Wednesday, January 10th, 2018



A picture of a Herrnhuter Stern/Moravian Star (Copyright: Frank Vincentz). In order to put a light bulb in, one has to remove a piece.

Exercise 33 (Symmetry group of the Herrnhuter Stern.)

What is the symmetry group of a complete Herrnhuter Stern?

Here complete means with no piece removed for the light bulb. A Herrnhuter Stern is composed of eighteen square and eight triangular cone-shaped points.

Exercise 34 (Compact-open topology and (co-)product) Let X, Y, Z be topological spaces.

1. Continuous maps $X \sqcup Y \to Z$ are in one-to-correspondence with pairs of maps $X \to Y$ and $Y \to Z$, so there is a canocical isomorphism

$$\mathrm{map}(X \sqcup Y, Z) \cong \mathrm{map}(X, Z) \times \mathrm{map}(Y, Z),$$

where $\mathrm{map}(X, Z)$ denotes the set of all continuous maps $X \to Z$. Proof that this isomorphism respects the compact-open topology, i.e. we have a homeomorphism:

$$\mathrm{Hom}(X \sqcup Y, Z) \cong \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z).$$

2. This statement is not true unless one requires additionaly properties (e.g. X being locally compact). For completeness the question nevertheless stays here. A counterexample is given in Addendum. Likewise the universal property of the product gives a canonical isomorphism

$$\mathrm{map}(X, Y \times Z) \cong \mathrm{map}(X, Y) \times \mathrm{map}(X, Z).$$

Proof that this as well induces a homeomorphism

$$\mathrm{Hom}(X, Y \times Z) \cong \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Z).$$

3. Do the previous statements extend to infite disjoint unions resp. infinite products?

Exercise 35 (Pushouts for topological spaces)

Suppose we are given topological spaces X, Y_1, Y_2 with (continuous) maps $\alpha_1 \colon X \to Y_1$ and $\alpha_2 \colon X \to Y_2$. A pushout of this consists of a topological space P together with maps $\varphi_1 : Y_1 \to P$ and $\varphi_2 \colon Y_2 \to P$ such that $\varphi_2 \circ \alpha_2 = \varphi_1 \circ \alpha_1$ and such that for all topological spaces Z with maps $\psi_1 : Y_1 \to Z$ and $\psi_2 \colon Y_2 \to Z$ with $\psi_2 \circ \alpha_2 = \psi_1 \circ \alpha_1$ there exists a unique map $h \colon P \to Z$ such that $h \circ \varphi_1 = \psi_1$ and $h \circ \varphi_2 = \psi_2$. More conveniently a pushout can be expressed with the following picture:

$$\begin{array}{ccccc} X & \xrightarrow{\alpha_1} & Y_1 & & \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \varphi_1} & \searrow{\scriptstyle \psi_1} & \\ Y_2 & \xrightarrow{\varphi_2} & P & & \\ & \searrow{\scriptstyle \psi_2} & & \overset{\exists! h}{\searrow} & \\ & & & & Z \end{array}$$

1. Proof that the pushout is isomorphic to

$$Y_1 \sqcup Y_2 / \sim$$

where $\sim$ is the relation given by $\alpha_1(x) \sim \alpha_2(x)$ for all $x \in X$.

Next we will calculate some concrete examples of pushouts. (Later, we will see that those particular examples are constructions of CW-complexes.)

2. $$\begin{array}{ccc} S^0 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ D^1 & & \end{array}$$

3. $$\begin{array}{ccc} S^0 & \xrightarrow{\mathrm{id}} & S^0 \\ \downarrow & & \\ D^1 & & \end{array}$$

4. $$\begin{array}{ccc} S^0 \sqcup S^0 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ D^1 \sqcup D^1 & & \end{array}$$

5. $$\begin{array}{ccc} S^1 & \xrightarrow{\mathrm{id}} & S^1 \\ \downarrow & & \\ D^2 & & \end{array}$$

6. $$\begin{array}{ccc} S^1 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ D^2 & & \end{array}$$

7. $$\begin{array}{ccc} S^1 \sqcup S^1 & \xrightarrow{\mathrm{id} \sqcup \mathrm{id}} & S^1 \\ \downarrow & & \\ D^2 \sqcup D^2 & & \end{array}$$

8. $$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \vee S^1 \\ \downarrow & & \\ D^2 & & \end{array}$$

f is described by going clockwise around the first circle, then clockwise around the second circle, then counter-clockwise around the first circle and then counter-clockwise around the second circle. A different way to describe this: Let $[a] \in \pi_1(S^1 \vee \mathrm{pt}, x_0)$, $[b] \in \pi_1(pt \vee S^1, x_0)$ represent generators (going around the circle in a specific direction. Then $f = aba^{-1}b^{-1}$.

9. $$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \vee S^1 \\ \downarrow & & \\ D^2 & & \end{array}$$

Here $f = abab^{-1}$.

10. $$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \downarrow & & \\ D^2 & & \end{array}$$

Here $f = a^2$.

Exercise 36 (Pushouts for groups) As above we can define a pushout for groups, where the G, H_1, H_2, P, L are groups and the maps are group homomorphisms:

$$\begin{array}{ccccc} G & \xrightarrow{\alpha_1} & H_1 & & \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \varphi_1} & \searrow{\scriptstyle \psi_1} & \\ H_2 & \xrightarrow{\varphi_2} & P & & \\ & \searrow{\scriptstyle \psi_2} & & \overset{\exists! h}{\searrow} & \\ & & & & L \end{array}$$

The pushout for groups is given by by $H_1 * H_2/N$ where N is the normal subgroup generated by $\alpha_1(g)\alpha_2(g^{-1})$ for all $g \in G$. In terms of generators and relations this is

$$\begin{aligned} \langle g \in H_1, h \in H_2 | h_1 \cdot_{H_1 * H_2} h_2 &= (h_2 \cdot_{H_1} h_2), h_1, h_2 \in H_1, \\ h_1 \cdot_{H_1 * H_2} h_2 &= (h_2 \cdot_{H_2} h_2),\ h_1, h_2 \in H_2, \\ \alpha_1(g) &= \alpha_2(g),\ g \in G \rangle, \end{aligned}$$

so the relations are given by the relations in H_1, the relations in H_2 and additionally the relations we need for the diagram to commute.

This construction can be arbitrarilly difficult to understand, however in particual examples, calculations are easier. Calculate the following pushouts:

1. $$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\ \downarrow & & \\ \{0\} & & \end{array}$$

   Where f is an arbitrary group homomorphisms.

2. $$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} * \mathbb{Z} \\ \downarrow & & \\ \{0\} & & \end{array}$$

   Here $f(g) = g_1 g_2^{-1}$, where g is a generator of $\mathbb{Z}$ and g_1, g_2 generate $\mathbb{Z} * \mathbb{Z}$.

3. $$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} * \mathbb{Z} \\ \downarrow & & \\ \{0\} & & \end{array}$$

   Here $f(g) = g_1 g_2 g_1^{-1} g_2^{-1}$, where g is a generator of $\mathbb{Z}$ and g_1, g_2 generate $\mathbb{Z} * \mathbb{Z}$.