Exercise Sheet for *Topology I*, 2017/18

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Exercise 34 (Compact-open topology and (co-)product) Let X, Y, Z be topological spaces.

1. Continuous maps $X \sqcup Y \to Z$ are in one-to-correspondence with pairs of maps $X \to Z$ and $Y \to Z$, so there is a canocical isomorphism

$$\mathrm{map}(X \sqcup Y, Z) \cong \mathrm{map}(X, Z) \times \mathrm{map}(Y, Z),$$

where $\mathrm{map}(X, Z)$ denotes the set of all continuous maps $X \to Z$. Proof that this isomorphism respects the compact-open topology, i.e. we have a homeomorphism:

$$\mathrm{Hom}(X \sqcup Y, Z) \cong \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z).$$

Solution Denote by $X \xrightarrow{i_1} X \sqcup Y \xleftarrow{i_2} Y$ the inclusions. The canonical isomorphism is given by the maps:

$$\varphi\colon \mathrm{map}(X \sqcup Y, Z) \to \mathrm{map}(X, Z) \times \mathrm{map}(Y, Z), \quad f \mapsto (f \circ i_1, f \circ i_2)$$

and

$$\psi\colon \mathrm{map}(X, Z) \times \mathrm{map}(Y, Z) \to \mathrm{map}(X \sqcup Y, Z), \quad (f, g) \mapsto (f \sqcup g).$$

($f \sqcup g$ denotes the unique map obtained by the universal property of the coproduct.)

By the universal property of the coproduct we see that those maps give an isomorphism. We need to check that they are continuous.

Let $K \subset X$ compact and $U \subset Z$ open. Denote by $V(K, U)$ the set of all continuous maps $f\colon X \to Z$ such that $f(K) \subset U$. The compact-open topology has those sets as a subbasis.

Note: Let X, Y be topological spaces and $\mathcal{S}, \mathcal{T}$ be a subbasis of X resp. Y then

$$\mathcal{S} \times \mathcal{T} = \{S \times T | S \in \mathcal{S}, T \in \mathcal{T}\}$$

is a subbasis of $X \times Y$.

φ is continuous: It suffices to show that following. For all $K \subset X, K' \subset Y$ compact and $U, U' \subset Z$ open the set

$$\varphi^{-1}(V(K, U) \times V(K', U'))$$

is open. But this is obvious. As

$$\varphi^{-1}(V(K, U) \times V(K', U')) = V(i_1(K), U) \cap V(i_2(K), U'),$$

which is open in $\mathrm{Hom}(X \sqcup Y, Z)$.

ψ is continuous: Every compact set in $X \sqcup Y$ has the form $K \sqcup K'$ for K compact in X and K' compact in Y. It suffices to show that for all $K \subset X, K' \subset Y$ compact and $U \subset Z$ open the set

$$\psi^{-1}(V(K \sqcup K', U))$$

is open. But this follows from

$$\psi^{-1}(V(K \sqcup K', U)) = V(K, U) \cap V(K', U).$$

2. **The one direction is only true if one assumes additional properties** Likewise the universal property of the product gives a canonical isomorphism

$$\mathrm{map}(X, Y \times Z) \cong \mathrm{map}(X, Y) \times \mathrm{map}(X, Z).$$

Proof that this as well induces a homeomorphism

$$\mathrm{Hom}(X, Y \times Z) \cong \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Z).$$

Solution Denote by $Y \xleftarrow{p_1} Y \times Z \xrightarrow{p_2} Z$ the prodjections. The canonical isomorphism is given by the maps:

$$\varphi\colon\ \mathrm{map}(X, Y \times Z) \to \mathrm{map}(X, Y) \times \mathrm{map}(X, Z), \quad f \mapsto (p_1 \circ f, p_2 \circ f)$$

and

$$\psi\colon\ \mathrm{map}(X, Y) \times \mathrm{map}(X, Z) \to \mathrm{map}(X, Y \times Z), \quad (f, g) \mapsto (f \times g).$$

($f \times g$ denotes the unique map obtained by the universal property of the product.)
By the universal property of the product we see that those maps give an isomorphism. We need to check that they are continuous.
φ is continuous: Let $K, K' \subset X$ be compact and $U \subset Y, U' \subset Z$ be open. We need to show that

$$\varphi^{-1}(V(K, U) \times V(K', U'))$$

is open. This follows from

$$\varphi^{-1}(V(K, U) \times V(K', U')) = V(K, U \times Z) \cap V(K', Y \times U').$$

Counterexample for ψ being continuous: The easiest example I can think of is unfortunately rather complicated. Let $X = \mathbb{N} \times \mathbb{N} \cup \{+\infty\} \cup \{-\infty\}$ along with the topology generated by the subbasis

$$\{U,\ [m, \infty) \times \mathbb{N} \cup \{+\infty\},\ \mathbb{N} \times [m, \infty) \cup \{-\infty\} | U \subset \mathbb{N} \times \mathbb{N}, m \in \mathbb{N}\}.$$

Those are the important properties of this set: X is compact but not locally compact. A set $K \subset X$ is compact if and only if one of the following is true: it is finite, it contains $+\infty$ and $-\infty$, it contains $+\infty$ and is bounded in the second coordinate or it contains $-\infty$ and is bounded in the first coordinate.
Now consider $Y, Z = \{0, 1, 2\}$ with the topology given by

$$\mathcal{O} = \{\emptyset, \{0, 1\}, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$

The map

$$f\colon X \to Y \times Z, \quad x \mapsto \begin{cases} (0, 0), & x = -\infty \\ (2, 2), & x = +\infty \\ (1, 1), & \text{otherwise} \end{cases}$$

is continuous (check!). It is contained in $V(X, U)$ for $U = \{1, 2\} \times \{1, 2\} \cup \{2, 3\} \times \{2, 3\}$ open.
For every $n \in \mathbb{N}$ the map

$$g_n\colon X \to Y \times Z, \quad x \mapsto \begin{cases} (0, 0), & x = -\infty \\ (2, 2), & x = +\infty \\ (0, 2), & x = (n, n) \\ (1, 1), & \text{otherwise} \end{cases}$$

is continuous (check!). It is not contained in $V(X, U)$.
We show that $\psi^{-1}V(X, U)$ is not open by showing that any open neighborhood of $\psi^{-1}(f)$ contains some $\psi^{-1}(g_n)$: To be more precise let

$$\psi^{-1}(f) \in \bigcup_{j \in J} \bigcap_{i=1}^{m} V(K_{i,j}, W_{i,j}) \times V(K'_{i,j}, W'_{i,j})$$

for $K_{i,j}, K'_{i,j} \subset X$ compact and $W_{i,j} \subset Y, W'_{i,j} \subset Z$ open. We claim that then $\psi^{-1}(g_n)$ is an element of this set. First we notice, that we may w.l.o.g. assume that J contains only one element. So

$$\psi^{-1}(f) \in \bigcap_{i=1}^{m} V(K_i, W_i) \times V(K'_i, W'_i).$$

Then $\psi^{-1}(f)$ is an element of each $V(K_i, W_i) \times V(K'_i, W'_i)$. It suffices to show that for some $n_2 \in \mathbb{N}$ depending on K_i, W_i, K'_i, W'_i all $\psi^{-1}(g_n)$ with $n > n_2$ are contained in $V(K_i, W_i) \times V(K'_i, W'_i)$. Then the finite intersection will still contain infinitely many $\psi^{-1}(g_n)$. Note that

$$\psi^{-1}(g_n) \in V(K_i, W_i) \times V(K'_i, W'_i) \quad \Leftrightarrow \quad p_1 \circ g_n \in V(K_i, W_i) \wedge p_2 \circ g_n \in V(K_i, W_i).$$

We consider all the possibilities for W_i:

- If $W_i = \emptyset$ then it follows that $K_i = \emptyset$ and $V(K_i, W_i) = \mathrm{Hom}(X, Y)$. This in turn implies that $p_1 \circ g_n \in V(K_i, W_i)$ for all $n \in \mathbb{N}$.
- Likewise $W_i = \{0, 1, 2\}$ implies $V(K_i, W_i) = \mathrm{Hom}(X, Y)$.
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  - If $0 \notin W_i$ then by $p_1 \circ f \in V(K_i, W_i)$ we have that $-\infty \notin K$. Then K_i is either finite or bounded in the second coordinate.
  - If $2 \notin W_I$ then K_i is bounded in the first coordinate.

  If K is bounded in some coordinate then $(n, n) \notin K_i$ for $n \geq n_0$ with $n_0 \in \mathbb{N}$ large enough. This in turn imlies that $p_1 \circ g_n \in V(K_i, W_i)$ for $n \geq n_0 \in \mathbb{N}$.

Since $Y = Z$ by the same argumentation we see that $p_2 \circ g_n \in V(K, W_i)$ for $n \geq n_1 \in \mathbb{N}$ with n_1 large enough.

This proofs our claim.

3. Do the previous statements extend to infite disjoint unions resp. infinite products?

Solution

As one could expect the second statement stays wrong. The statement about infinite disjoint unions extends. The reason being that a compact set in $\bigsqcup_{i \in I} X_i$ may only intersect finitely many X_i. Likewise an open set in a product $\prod_{i \in I} \mathrm{Hom}(X_i, Z)$ is of the form $\prod_{i \in I} U_i$ for $U_i \subset X$ open, where $U_i = X$ except for finitely many U_i.