Midterm Quiz for *Topology I*, 2017/18

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Tag: ______________________________

This quiz is made to give you and us some feedback, how well notions are understood. There is no need for your name, but you should be able to identify your quiz.

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1. Given a topological space X and a subset $A \subset X$. Let $U \subset A$ be **open** in A with respect to the subspace topology. Which conditions will suffice to make U open in X.
   - ◯ No farther conditions needed.
   - ◯ A is closed.
   - √ A **is open.**
   - ◯ A compact.
   - ◯ A compact and X Hausdorff.

2. Given a topological space X and a subset $A \subset X$. Let $U \subset A$ be **closed** in A with respect to the subspace topology. Which conditions will suffice to make U closed in X.
   - ◯ No farther conditions needed.
   - √ A **is closed.**
   - ◯ A is open.
   - ◯ A compact.
   - √ A **compact and** X **Hausdorff.**

3. We have seen some properties of spaces and examples where those properties are preserved and were they are not preserved.

   (a) Which of the following constructions preserve T_1?

   √ **Finite Product.** √ **Subspace.** ◯ Quotient. √ **Infinite Disjoint Union.**

   (b) Which of the following constructions preserve Hausdorff?

   √ **Finite Product.** √ **Subspace.** ◯ Quotient. √ **Infinite Disjoint Union.**

   (c) Which of the following constructions preserve compactness?

   √ **Finite Product.** ◯ Subspace. √ **Quotient.** ◯ Infinite Disjoint Union.

   (d) Which of the following constructions preserve path-connectness?

   √ **Finite Product.** ◯ Subspace. √ **Quotient.** ◯ Infinite Disjoint Union.

   (e) Suppose we start with (a) metric space(s) with induced topology. Which of the following constructions give rise to some induced metric, that induces the correct topology on the construction?

   √ **Finite Product.** √ **Subspace.** ◯ Quotient. √ **Infinite Disjoint Union.**

4. Given some topological spaces X and Y and a set map $f: X \to Y$. What is sufficient to check for continuity?

   ◯ For every $U \subset Y$ open it holds that $f^{-1}(U)$ is contained in some open set.

   √ **For every $B \subset Y$ we have $f^{-1}(B^\circ) \subset f^{-1}(B)^\circ$.**

   ◯ For every $B \subset Y$ we have $f^{-1}(B)^\circ \subset f^{-1}(B^\circ)$.

   √ **For every $U \subset Y$ open and every $x \in U$ we can find some $x \in V \subset U$ such that V is open with respect to the subspace topology and $f^{-1}(V)$ is open.**

   ◯ For every $U \subset Y$ open there is some $V \subset U$ open in Y such that $f^{-1}(V)$ is open.

   ◯ For all $x, y \in Y$ there is $U \subset Y$ open with $x \in U \not\ni y$ and such that $f^{-1}(U)$ is open.

   ◯ For all $x, y \in Y$ we can find some $A \subset Y$ closed with $x \in A \not\ni y$ such that $f^{-1}(A)$ is closed.

   √ **X has discrete topology.**

   ◯ Y has discrete topology.

   ◯ X has indiscrete topology.

   √ **Y has indiscrete topology.**

5. Which of the following statements is true? Which are false?

   (a) Let $f: X \to Y$ be an injective **open** continuous map and let X be compact. Then Y is compact.

   ◯ True. √ **False.**

   (b) Let $f: X \to Y$ be an injective **closed** continuous map and let X be compact. Then Y is compact.

   ◯ True. √ **False.**

   (c) Let $f: X \to Y$ be an injective **open** continuous map and let Y be compact. Then X is compact.

   ◯ True. √ **False.**

   (d) Let $f: X \to Y$ be an injective **closed** continuous map and let Y be compact. Then X is compact. √ **True.** ◯ False.

6. Let X be a space and let $P \subset X$ be a path-connected compontent. What do we know?
   - $\surd$ **P is connected.**
   - $\surd$ **P is path-connected.**
   - ◯ P is open.
   - ◯ P is closed.

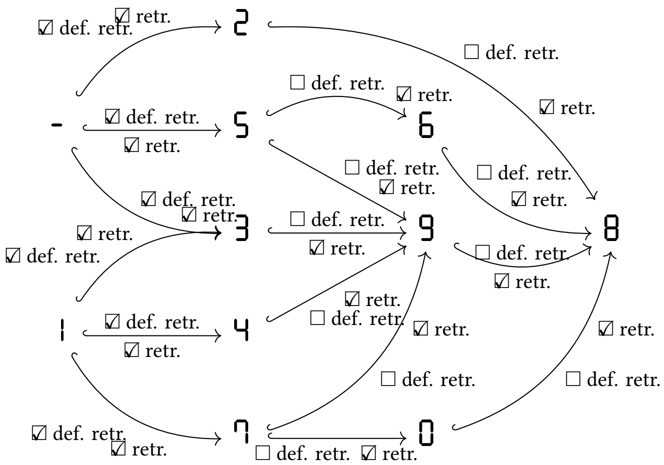
7. Let X be a space and let $P \subset X$ be a connected compontent. What do we know?
   - $\surd$ **P is connected.**
   - ◯ P is path-connected.
   - ◯ P is open.
   - $\surd$ **P is closed.**

8. Consider the space $Y = [-1, 1]$ as a subspace of $\mathbb{R}$ with Euclidean topology. Which of the following sets are **open** in Y, which are **open** in $\mathbb{R}$?
   (a) $A = \{x \mid \frac{1}{2} < |x| < 1\}$. $\surd$ **Open in Y.** $\surd$ **Open in $\mathbb{R}$.**
   (b) $B = \{x \mid \frac{1}{2} < |x| \leq 1\}$. $\surd$ **Open in Y.** ◯ Open in $\mathbb{R}$.
   (c) $C = \{x \mid \frac{1}{2} \leq |x| < 1\}$. ◯ Open in Y. ◯ Open in $\mathbb{R}$.
   (d) $D = \{x \mid \frac{1}{2} \leq |x| \leq 1\}$. ◯ Open in Y. ◯ Open in $\mathbb{R}$.
   (e) $E = \{x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}\}$. $\surd$ **Open in Y.** $\surd$ **Open in $\mathbb{R}$.**

9. Consider the space $Y = (-1, 1) \times [-1, 1]$ as a subset of $\mathbb{R}^2$ with the Euclidean topology. Which of the following sets are **closed** in Y, which are **closed** in $\mathbb{R}^2$?
   (a) $A = \{(x, y) \in Y \mid x^2 + y^2 = 1\}$. $\surd$ **Closed in Y.** ◯ Closed in $\mathbb{R}^2$.
   (b) $B = \{(x, y) \in Y \mid x^2 + y^2 = \frac{1}{2}\}$. $\surd$ **Closed in Y.** $\surd$ **Closed in $\mathbb{R}^2$.**
   (c) $C = \{(x, y) \in Y \mid x^2 + y^2 \leq 1\}$. $\surd$ **Closed in Y.** ◯ Closed in $\mathbb{R}^2$.
   (d) $D = \{(x, y) \in Y \mid 2x^2 + y^2 \leq 1\}$. $\surd$ **Closed in Y.** $\surd$ **Closed in $\mathbb{R}^2$.**
   (e) $E = \{(x, y) \in Y \mid x^2 + 2y^2 \leq 1\}$. $\surd$ **Closed in Y.** ◯ Closed in $\mathbb{R}^2$.
   (f) $E = \{(x, y) \in Y \mid x + y > 1\}$. ◯ Closed in Y. ◯ Closed in $\mathbb{R}^2$.
   (g) $F = \{(x, y) \in Y \mid 2x + y \geq 1\}$. $\surd$ **Closed in Y.** ◯ Closed in $\mathbb{R}^2$.

10. Let $A, B, (C_i)_{i \in I} \subset X$. Which inclusions hold?
   (a) $\overline{A \cap B}$ $\surd \subset$ $\bigcirc \supset$ $\overline{A} \cap \overline{B}$
   (b) $\overline{A \cup B}$ $\surd \subset$ $\surd \supset$ $\overline{A} \cup \overline{B}$
   (c) $\overline{A \times B}$ $\surd \subset$ $\surd \supset$ $\overline{A} \times \overline{B}$
   (d) $\overline{A \setminus B}$ $\bigcirc \subset$ $\surd \supset$ $\overline{A} \setminus \overline{B}$
   (e) $\overline{\bigcap_{i \in I} C_i}$ $\surd \subset$ $\bigcirc \supset$ $\bigcap_{i \in I} \overline{C_i}$
   (f) $\overline{\bigcup_{i \in I} C_i}$ $\bigcirc \subset$ $\surd \supset$ $\bigcup_{i \in I} \overline{C_i}$
   (g) $\overline{\prod_{i \in I} C_i}$ $\bigcirc \subset$ $\surd \supset$ $\prod_{i \in I} \overline{C_i}$

11. Remember the digits your old alarm clock produced: We consider each digit as a topological space with thin lines and connected such that 1 and 0 are homeomorphic to $I = [0, 1]$ resp. S^1. Those spaces come with canonical inclusions. Mark for each inclusion if this is the inclusion of a retract and if this is the inclusion of a deformation retract:



Addendum from Monday, December 11th

Todays tutorial discoverd four mistakes in the quiz. Two in question 5 and two in question 10. In question 5 (a) and (b) the image of a compact space needs to be compact, but the map was never claimed to be surjective. E.g. the inclusion

$$[0,1] \hookrightarrow \mathbb{R}$$

is closed injective, but $\mathbb{R}$ is not compact. Likewise

$$(0,1) \hookrightarrow (0,1) \sqcup \mathbb{R}$$

is open injective, but $(0,1) \sqcup \mathbb{R}$ is not compact.
As of question 10, it is actually true, that $\overline{A} \backslash \overline{B} \subset \overline{A \backslash B}$. We have to show, that if $x \in \overline{A}$ then $x \in \overline{B}$ or $x \in \overline{A \backslash B}$, i.e. $\overline{A} \subset \overline{B} \cup \overline{A \backslash B}$. But this follows from $\overline{B} \cup \overline{C} = \overline{B \cup C}$ for $C = A \backslash B$.
Also $\overline{\bigcap_{i \in I} C_i} \subset \bigcap_{i \in I} \overline{C_i}$. This is simply because $\bigcap_{i \in I} \overline{C_i}$ is a closed set that contains $\bigcap_{i \in I} C_i$ (because $C_i \subset \overline{C_i}$) as well as it's closure.