Exercise Sheet for *Topology II*, 18

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Sheet 1 Discussion: Wednesday, April 25th, 2018

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

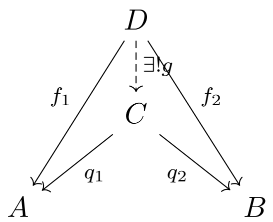
The formal definition (without pictures) of a natural transformation can be hard to grasp. Basically, a natural transformation is something that makes the above diagram commmute. More formal: Let F, G be functors $\mathcal{C} \to \mathcal{D}$. A natural transformation $\alpha \colon F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that for every $B \in \mathcal{C}$ and any $f \colon \mathcal{C}(A, B)$ the above diagram commutes.

Exercise 1 Show that a map in a category can have at most one inverse, i.e., let $A, B \in \mathcal{C}$ and $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$. Further, let $g, h \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ such that $g \circ f = h \circ f = \mathrm{id}_A$. Then $g = h$.

Exercise 2 Let $\mathcal{C}, \mathcal{D}$ be categories. There is only one sensible way to define $\mathcal{C} \times \mathcal{D}$. Define it!

Exercise 3 Prove that $\mathcal{C} \times \mathcal{D}$ has the universal property of a product in the category of categories. The category of categories consists of categories as objects and functors as morphisms.

The fundamental property of a product: Let $A, B \in \mathcal{C}$. An object $C \in \mathcal{C}$ with maps $p_1 \in \mathrm{Mor}(C, A)$ and $p_2 \in \mathrm{Mor}(C, B)$ is called the product of A and B if for every $D \in \mathcal{D}$ with maps $f_1 \in \mathrm{Mor}(D, A)$, $f_2 \in \mathrm{Mor}(D, B)$ there exists a unique morphism $g \in \mathrm{Mor}(D, C)$ such that $q_1 \circ g = f_1$ and $g_1 \circ g = f_2$:



Exercise 4 Prove that functors preserve isomorphisms, i.e., let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and let $f \in \mathrm{Mor}(A, B)$ be an isomorphism, then $F(f)$ is also an isomorphism.

Does this hold for contravariant functors as well?

Exercise 5 Let G be a group. Consider G as a category with one object and the group elements as morphisms. What is a subcategory of G?

Exercise 6 Let G be a group regarded as category.

1. Prove that a functor from G to the category of topological spaces is a (left) G-space.
   A left G-space is a topological space with a left-G-action such that this action is continuous (for all $g \in G$ the map $x \mapsto g \circ x$ is continuous).
2. What are natural transformations betweens G-spaces?