Exercise Sheet for *Topology II*, 18

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Sheet 2 Discussion: Wednesday, May 2nd, 2018

$$0 \to \mathbb{Z} \xrightarrow{x \mapsto 2x} \mathbb{Z} \xrightarrow{\text{proj}} \mathbb{Z}/2\mathbb{Z} \to 0$$

This is a short exact sequence. Long and short exact sequences are an important tool and a good way to present a lot of information in a small diagram.
An exact sequence is a sequence of maps (finite or infinite)

$$\dots G_n \xrightarrow{f_n} G_{n+1} \xrightarrow{f_{n+1}} G_{n+2} \xrightarrow{f_{n+2}} \dots$$

such that $\operatorname{im}(f_n) = \ker(f_{n+1})$ for all n where f_n and f_{n+1} are defined.

The goal this session is to show the fundamental theorem of abelian groups, which is the last exercise. Throughout this exercise we will deal with additive abelian groups with neutral element 0.

Exercise 7 Let $(x_1, \dots, x_k)$ generate a finite abelian group A. Suppose $c_1, \dots, c_k \in \mathbb{Z}$ such that $\gcd(c_1, \dots, c_k)$. Then there exist $(y_1, \dots, y_k)$ that generate A such that $y_1 = c_1x_1 + \dots + c_kx_k$.

Hint: Induction on $\sum |c_k|$.

Exercise 8 Show that any finitely generated abelian group A has a basis. That is A is generated by some $(y_1, \dots, y_k)$ where $y_i \neq 0$ for all i such that $\sum_{i=1}^k \lambda_i y_i = 0$ implies $\lambda_i y_i = 0$ for all i.

Exercise 9 Let $n \in \mathbb{N}$ and let $n_1, n_2 \in \mathbb{N}$ such that $\gcd(n_1, n_2) = 1$ and $n_1n_2 = n$. Show that the cyclic group

$$\mathbb{Z}/n\mathbb{Z}$$

is isomorphic to

$$\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}.$$

Hint: There exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha n_1 + \beta n_2 = \gcd(n_1, n_2)$ (Bézout's identity).

Exercise 10 (Existence) Conclude: Let A be a finitely generated abelian group. There exists k, l and prime powers $p_1^{e_1} \leq \dots \leq p_l^{e_l}$ such that

$$A \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{e_i}\mathbb{Z}.$$

Exercise 11 (Uniqueness) Let $k, l, \kappa, \lambda \in \mathbb{N}$ and let $p_1^{e_1} \leq \dots \leq p_l^{e_l}$ and $q_1^{f_1} \leq \dots \leq q_l^{f_\lambda}$ be prime powers.

Show that

$$\mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{e_i}\mathbb{Z} \cong \mathbb{Z}^\kappa \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/q_i^{f_i}\mathbb{Z}$$

implies $k = \kappa, l = \lambda$ and $p_i^{e_i} = q_i^{f_i}$ for all $i = 1, \dots, l$.