Exercise Sheet for *Topology II*, 18

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Sheet 3 Discussion: Wednesday, May 9th, 2018

Exercise 12 (**The** product, **the** coproduct, ...)

1. Let $\mathcal{C}$ be a category and A, B be objects of $\mathcal{C}$. Assume that their product and coproduct exist. Denote by $A \sqcup B$ the coproduct and $A \times B$ their product. Show that they are unique up to canocial isomorphism. (Use the fundamental property of a product and of a coproduct.)

   In fact any limit or colimit is unique up to canonical isomorphism.
2. In which categories do (co-)products exist? How are they constructed?

Exercise 13 Assume that $\mathcal{C}$ is a category with products and coproducts. Remember the construction of $\mathcal{C} \times \mathcal{C}$. Show that product and coproduct define functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Exercise 14 Assume that a category $\mathcal{C}$ has product and coproduct. Assume in addition that $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is equipped with the structure of an abelian group for all A, B objects of $\mathcal{C}$, which is distributive, i.e. $(f + f') \circ g = f \circ g + f' \circ g$ and $f \circ (g + g') = f \circ g + f \circ g'$. Show that there is a unique map $\kappa \colon A \sqcup B \to A \times B$ such that

$$\begin{aligned} p_A \circ \kappa \circ i_A = \mathrm{id}_A, \quad p_B \circ \kappa \circ i_B = \mathrm{id}_B \\ p_A \circ \kappa \circ i_B = 0, \quad p_B \circ \kappa \circ i_A = 0 \end{aligned}$$

and the map κ is an isomorphism. (i_A, i_B denote the inclusion maps $A, B \hookrightarrow A \sqcup B$, p_A, p_B denote the projections $A \times B \to A, B$.)

For which categories do you know of an abelian distributive structure on $\mathrm{Mor}_{\mathcal{C}}(A, B)$?

Exercise 15 (Homology of a complex) A $\mathbb{Z}$-complex consists of abelian groups C_n for $n \in \mathbb{Z}$ and maps $d_n \colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. We can then calculate homology of the complex:

$$H_n(C) := \ker(d_n)/\operatorname{im}(d_{n+1})$$

1. Construct a complex such that $H_0(C) \cong \mathbb{Z}$ $H_n(C) = 0$ for $n \neq 0$.
2. Construct a complex such that $H_n(C) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 1$, $H_0(C) = \mathbb{Z}$ and $H_n(C) = 0$ for $n < 0$.

Exercise 16 (Snake Lemma)

1. Recall: Let R be a commutative ring and let M, N be R-modules. Let $f \colon M \to N$ be an R-module homomorphism then there exists an exact sequence

   $$0 \to \ker(f) \to M \to N \to \operatorname{coker}(f) \to 0,$$

   where $\ker(f) = \{x \in M | f(x) = 0\}$ and $\operatorname{coker}(f) = N/\operatorname{im}(f)$.

2. Let A, B, C, A', B', C' be R-modules. Suppose the following commutative diagram has exact rows:

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

Prove that there exists an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h).$$