Exercise Sheet for *Topology II*, 18

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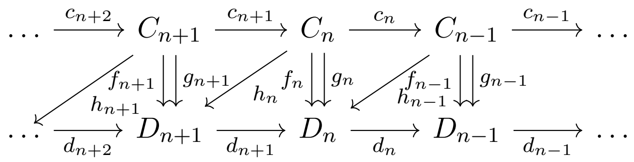
Sheet 4 Discussion: Wednesday, May 16th, 2018



Let $(C_\bullet, c_\bullet)$ and $(D_\bullet, d_\bullet)$ be chain $\mathbb{Z}$-chain complexes and $f, g\colon C_\bullet \to D_\bullet$ be chain maps. A chain homotopy between f and g is a a collection of group homomorphisms $h_n\colon C_n \to D_{n+1}$ for each $n \in \mathbb{Z}$ such that

$$f_n - g_n = h_{n-1} \circ c_n + d_{n+1} \circ h_n.$$

Exercise 17 (Product and Coproduct of chain complexes) Consider the category of $\mathbb{Z}$-chain complexes consisting of $\mathbb{Z}$-chain complexes as objects and chain maps as maps. Prove that this category has product and coproduct.

Exercise 18 Prove that $H_n(C)$ defines a functor from $\mathbb{Z}$-chain complexes to abelian groups for all $n \in \mathbb{Z}$.

Suppose $f_\bullet, g_\bullet\colon C_\bullet \to D_\bullet$ are chain maps that are homotopic (so there is a chain homotopy from f to g). Prove that $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.

There is a homotopy category of $\mathbb{Z}$-chain complexes, where chain maps which are homotopic are considered to be the same. Conclude that $H_n(C)$ defines a functor from the homotopy category of $\mathbb{Z}$-chain complexes to the category of abelian groups.

Exercise 19 Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups and let D be an abelian group.

What part of

$$0 \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \to 0$$

is exact?

What part of

$$0 \to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C) \to 0$$

is exact?

What part of

$$0 \to A \otimes D \to B \otimes D \to C \otimes D \to 0$$

is exact (google tensor product, if you don't know what it is).

Give counterexamplexes for those parts, that are not exact.