Exercise Sheet for *Topology II*, 18

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Sheet 5 Discussion: Wednesday, May 23rd, 2018

Exercise 20 (More diagram chasing) Let R be a ring and consider the following commuative diagram of R-modules:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & C' & \xrightarrow{\gamma_1} & C & \xrightarrow{\gamma_2} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Assume it is known, that all the rows and columns except for

$$0 \to B' \xrightarrow{\beta_1} B \xrightarrow{\beta_2} B'' \to 0$$

are exact. Assume further that $\operatorname{im}\beta_1 \subset \ker\beta_2$ or that $\ker\beta_2 \subset \operatorname{im}\beta_1$. Prove that all rows and columns are exact!

Exercise 21 Let $h_* = (h_n, \partial_n^h)_{n\in\mathbb{Z}}$ be a homology theory satisfying axioms (1)–(4) and let Y be a topological space. Show that for suitably defined ∂_n^k we obtain a new homology theory k_* satisfying axiom (1)–(3) be setting

$$k_n(X, A) = h_n(X, A) \oplus h_n(X \times Y, A \times Y).$$

If h_* satisfies axiom (5) (additivity), so does k_*. If Y is contractible, then k_* satifies (4) (so k_* is ordinary).

Exercise 22 Let h_* be a homology theory satisfying axioms (1)–(5) with coefficients in $\mathbb{Z}$. Let X be a finite graph with X_0 the set of vertices.

1. For all $x \in X$ and all $n \in \mathbb{Z}$ show that $h_n(X, X\backslash\{x\})$ is a finitely generated free $\mathbb{Z}$-module and define the valence of x as

$$\operatorname{val}(x) = \operatorname{rk}_{\mathbb{Z}} h_1(X, X\backslash\{x\}) + 1.$$

2. If $x \notin X_0$ then $\operatorname{val}(x) = 2$.
3. A homeomorphism $f\colon X \to Y$ is valence preserving, i.e., it sends points of valence k to points of valence k.