

# Exercise Sheet for *Topology II*, 18

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Sheet 5

Discussion: Wednesday, May 23rd, 2018

**Exercise 20** (More diagram chasing) Let  $R$  be a ring and consider the following commutative diagram of  $R$ -modules:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \longrightarrow 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' \\
 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \longrightarrow 0 \\
 & & \downarrow g' & & \downarrow g & & \downarrow g'' \\
 0 & \longrightarrow & C' & \xrightarrow{\gamma_1} & C & \xrightarrow{\gamma_2} & C'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Assume it is known, that all the rows and columns except for

$$0 \rightarrow B' \xrightarrow{\beta_1} B \xrightarrow{\beta_2} B'' \rightarrow 0$$

are exact. Prove that all rows and columns are exact!

**Exercise 21** Let  $h_* = (h_n, \partial_n^h)_{n \in \mathbb{Z}}$  be a homology theory satisfying axioms (1)–(4) and let  $Y$  be a topological space. Show that for suitably defined  $\partial_n^k$  we obtain a new homology theory  $k_*$  satisfying axiom (1)–(3) by setting

$$k_n(X, A) = h_n(X, A) \oplus h_n(X \times Y, A \times Y).$$

If  $h_*$  satisfies axiom (5) (additivity), so does  $k_*$ . If  $Y$  is contractible, then  $k_*$  satisfies (4) (so  $k_*$  is ordinary).

**Exercise 22** Let  $h_*$  be a homology theory satisfying axioms (1)–(5) with coefficients in  $\mathbb{Z}$ . Let  $X$  be a finite graph with  $X_0$  the set of vertices.

1. For all  $x \in X$  and all  $n \in \mathbb{Z}$  show that  $h_n(X, X \setminus \{x\})$  is a finitely generated free  $\mathbb{Z}$ -module and define the valence of  $x$  as

$$\text{val}(x) = \text{rk}_{\mathbb{Z}} h_1(X, X \setminus \{x\}) + 1.$$

2. If  $x \notin X_0$  then  $\text{val}(x) = 2$ .
3. A homeomorphism  $f: X \rightarrow Y$  is valence preserving, i.e., it sends points of valence  $k$  to points of valence  $k$ .